Basic relative invariants of homogeneous convex cones

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Background

$\Omega \subset V$: homogeneous convex cone
$\Delta_1(x), \dots, \Delta_r(x)$: basic relative invariants of Ω

$$\Omega = \{x \in V;\ \Delta_1(x) > 0, \dots, \Delta_r(x) > 0\}.$$

Theorem (Vinberg 1963)

Homogeneous convex domains	$\Leftrightarrow$	Clans
Homogeneous convex cones	$\Leftrightarrow$	Clans with unit

$R_x y := y \triangle x$: right multiplication operator

$$\mathrm{Det}\, R_x = \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad (n_j \geq 1).$$

Contents

1. Preliminary
   clans, basic relative invariants, representations, ...
2. Inductive structure of a clan and of the basic relative invariants
3. Introduce the multiplier matrix and ε-representations
4. Explicit formula of the basic relative invariants

Contents

Clans (compact normal left symmetric algebras)

V : finite-dimensional real vector space
$\triangle$: bilinear product in V

Definition

$(V, \triangle)$ is a clan $\Leftrightarrow$ the following three conditions are satisfied:

(C1) $[L_x, L_y] = L_{x \triangle y - y \triangle x}$, (left symmetric algebra)

(C2) $\exists s \in V^*$ s.t. $s(x \triangle y)$ is an inner product, (compactness)

(C3) L_x has only real eigenvalues. (normality)

($L_x y := x \triangle y$: left multiplication operator)

In general, clans are
- non-associative,
- non-commutative,
- no unit element.

Examples

$V = \mathbf{Herm}(r, \mathbb{K}), \quad (\mathbb{K} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H}).$

- $x \triangle y := \underline{x}\, y + y(\underline{x})^* \ (x, y \in V)$.

$$\underline{x} := \begin{pmatrix} \frac{1}{2}x_{11} & 0 & \cdots & 0 \\ x_{21} & \frac{1}{2}x_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x_{r1} & x_{r2} & \cdots & \frac{1}{2}x_{rr} \end{pmatrix}.$$

Normal decomposition

- V: clan with unit element e_0,
- $c_1, \ldots, c_r$: complete system of orthogonal primitive idempotents $(c_i \triangle c_j = \delta_{ij} c_i,\ c_1 + \cdots + c_r = e_0)$
- Normal decomposition: $V = \bigoplus_{1 \le j \le k \le r} V_{kj}$, where

$$\begin{cases} V_{jj} = \mathbb{R} c_j \quad (j = 1, \ldots, r), \\ V_{kj} = \left\{ x \in V;\ L_{c_i} x = \frac{1}{2}(\delta_{ij} + \delta_{ik}) x,\ R_{c_i} x = \delta_{ij} x \right\}. \end{cases}$$

In the case of $V = \mathrm{Sym}(r, \mathbb{R})$,

- $c_j = E_{jj}$,
- $V_{kj} = \mathbb{R}(E_{kj} + E_{jk})$.

Basic relative invariants

- $\mathfrak{h} := \{L_x;\ x \in V\}$ (split solvable Lie algebra).
- $H := \exp \mathfrak{h}$.
- $\Omega := H \cdot e_0 \Rightarrow$homogeneous cone.
  In particular, $H \curvearrowright \Omega$: simply transitively.

Definition.

1. $f : \Omega \to \mathbb{R}$: relatively H-invariant
   $\Leftrightarrow \exists \chi : H \to \mathbb{R}$: 1-dim. representation s.t. $f(hx) = \chi(h) f(x)$.
2. $\Delta_j(x)$: relatively H-invariant irreducible polynomials
   $(j = 1, \ldots, r)$
   $\Rightarrow$the basic relative invariants

Remark. **(Ishi 2001, Ishi–Nomura 2008)**
$\forall p(x)$: relatively H-invariant polynomial

$$\Rightarrow p(x) = (\mathrm{const}) \Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \ (m_1, \ldots, m_r \in \mathbb{Z}_{\geq 0}).$$

If $p(x) = \mathrm{Det}\, R_x$, then we have $m_k \geq 1$ $(k = 1, \ldots, r)$.

Dual clan

Definition

$(V, \triangledown)$: the dual clan of V

$$\langle\, x \triangledown y \,|\, z \,\rangle = \langle\, y \,|\, x \vartriangle z \,\rangle \quad (x, y, z \in V).$$

$$\begin{array}{rccc} \text{homogeneous cone} & \Omega & \longleftrightarrow & \Omega^* \\ & \updownarrow & \text{dual} & \updownarrow \\ \text{clan} & (V, \vartriangle) & \longleftrightarrow & (V, \triangledown) \end{array}$$

- Relation between $\vartriangle$ and $\triangledown$:

$$x \vartriangle y + x \triangledown y = y \vartriangle x + y \triangledown x.$$

Examples

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- Corresponding cone: $\Omega = \{x \in V;\ \text{positive definite}\}$.
- basic relative invariants: $\Delta_k(x) = \det^{(k)}(x)$.
- $\mathrm{Det}\, R_x = \Delta_1(x)^d \cdots \Delta_{r-1}(x)^d \Delta_r(x)\ (d = \dim \mathbb{K})$.
- Dual clan product:

$$x \bigtriangledown y = (\underline{x})^* y + y\, \underline{x} \quad (x, y \in V).$$

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Representations of clans

E: a real Euclidean vector space with $\langle \cdot | \cdot \rangle_E$

Definition

Let $\varphi\colon V \to \mathcal{L}(E) = \{\text{Linear maps on } E\}$.
(φ, E): a selfadjoint representation of the dual clan $(V, \triangledown)$:

- $\varphi(x)^* = \varphi(x)$ and $\varphi(e_0) = \mathrm{id}_E$,
- $\varphi(x \triangledown y) = \overline{\varphi}(x)\varphi(y) + \varphi(y)\underline{\varphi}(x)$,

where $\underline{\varphi}(x)$ (resp. $\overline{\varphi}(x)$) is lower (resp. upper) triangular part of $\varphi(x)$.

i.e. $\varphi\colon (V, \triangledown) \to (\mathrm{Sym}(E), \triangledown)$ is a homomorphism of a clan.

Definition. $Q\colon E \times E \to V$: bilinear map associated with φ:

$$\langle Q(\xi, \eta) \,|\, x \rangle = \langle \varphi(x)\xi \,|\, \eta \rangle_E \quad (\xi, \eta \in E,\ x \in V).$$

$Q[\xi] := Q(\xi, \xi)$ and $Q[E] := \{Q[\xi];\ \xi \in E\}$.

Contents

Inductive structure of V

Ω : Homogeneous cone
V : clan associated with Ω
$V = \bigoplus_{j \le k} V_{kj}$: normal decomposition

$$\text{Put} \quad E = \bigoplus_{k \ge 2} V_{k1}, \quad W = \bigoplus_{2 \le j \le k \le r} V_{kj}.$$

Note that W is a subclan of V.
V is decomposed as

$$V = V_{11} \oplus E \oplus W = \begin{pmatrix} \mathbb{R}c_1 & {}^tE \\ E & W \end{pmatrix}.$$

We denote general elements x of V by

$$x = \lambda c_1 + \xi + w \quad (\lambda \in \mathbb{R},\ \xi \in E,\ w \in W).$$

Inductive structure of V

Proposition

Define a linear map $\varphi\colon W \to \mathcal{L}(E)$ by

$$\varphi(w)\xi := \xi \mathbin{\triangledown} w \quad (w \in W,\ \xi \in E).$$

Then (φ, E) is a selfadjoint representation of $(W, \triangledown)$.

With respect to this decomposition, the multiplication is described as

$$x \vartriangle y = (\lambda\mu)c_1 + (\mu\xi + \tfrac{1}{2}\lambda\eta + \underline{\varphi}(w)\eta) + (Q(\xi,\eta) + w \vartriangle v),$$

where $y = \mu c_1 + \eta + v$.

Calculate $\mathrm{Det}\, R_x$ and express by using $\mathrm{Det}\, R_w^W$

Right multiplication operators

$\boldsymbol{R}$:Right multiplication operator of $\boldsymbol{V}$

Then we have $$\boldsymbol{R_{\lambda c_1+\xi+w}} = \begin{pmatrix} \boldsymbol{\lambda} & \mathbf{0} & \mathbf{0} \\ \frac{1}{2}\boldsymbol{\xi} & \boldsymbol{\lambda}\mathbf{id}_{\boldsymbol{E}} & \boldsymbol{R_\xi} \\ \mathbf{0} & \boldsymbol{R_\xi} & \boldsymbol{R^W_w} \end{pmatrix},$$

where $\boldsymbol{R^W}$ is right multiplication operator of $\boldsymbol{W}$.

Proposition

$$\mathbf{Det}\, \boldsymbol{R_{\lambda c_1+\xi+w}} = \boldsymbol{\lambda}^{\mathbf{1+dim}\,\boldsymbol{E}-\mathbf{dim}\,\boldsymbol{W}}\, \mathbf{Det}\, \boldsymbol{R^W_{\lambda w - \frac{1}{2}Q[\xi]}}$$

Right multiplication operators

The basic relative invariants of V are exhausted by

$$\begin{cases} \lambda, \\ \text{irreducible factors of } \Delta_j^W(\lambda w - \frac{1}{2}Q[\xi]) \quad (j = 2, \dots, r), \end{cases}$$

where $\Delta_2^W(w), \dots, \Delta_r^W(w)$ are the basic relative invariants of W.

Theorem

$\Delta_1(x), \dots, \Delta_r(x)$: the basic relative invariants of V.
There exist non-negative integers $\alpha_2, \dots, \alpha_r$ s.t.

$$\Delta_j(x) = \begin{cases} \lambda & (j = 1), \\ \lambda^{-\alpha_j}\Delta_j^W(\lambda w - \frac{1}{2}Q[\xi]) & (j = 2, \dots, r). \end{cases}$$

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Contents

In order to determine α_j, we introduce

1. multiplier matrix,
2. ε-representation.

Multiplier matrix

- H: split solvable (= lower triangular) Lie group
  $h_{jj} > 0$: diagonal component of $h \in H$

- f: relatively H-invariant ($\exists \chi$ s.t. $f(hx) = \chi(h)f(x)$)

$$\Rightarrow \exists \tau_j \in \mathbb{R} \text{ s.t. } \chi(h) = (h_{11})^{2\tau_1} \dots (h_{rr})^{2\tau_r}.$$

$\underline{\tau} := (\tau_1, \dots, \tau_r)$: multiplier of f.

Multiplier matrix

Definition

$\underline{\sigma}_j = (\sigma_{j1}, \ldots, \sigma_{jr})$: multiplier of Δ_j

$$(\Delta_j(hx) = (h_{11})^{2\sigma_{j1}} \cdots (h_{rr})^{2\sigma_{jr}} \Delta_j(x))$$

$$\sigma : \text{the multiplier matrix} \Leftrightarrow \sigma = \begin{pmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_r \end{pmatrix} = (\sigma_{jk})_{1 \le j,k \le r}$$

Remark. (**Ishi 2001**)

- σ is (lower) triangular,
- $\sigma_{jj} = 1$.

$$\text{i.e.} \quad \sigma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \sigma_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \sigma_{r1} & \sigma_{r2} & \cdots & 1 \end{pmatrix}.$$

ε-representation

(φ, E): representation of $(V, \triangledown)$
$\varepsilon = {}^t(\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r$, $c_\varepsilon := \varepsilon_1 c_1 + \cdots + \varepsilon_r c_r$.

Definition

$$(\varphi, E): \ \varepsilon\text{-representation} \Leftrightarrow Q[E] = \overline{H \cdot c_\varepsilon}.$$

Remark. Put $\mathcal{O}_\varepsilon := H \cdot c_\varepsilon$. Then one has

$$\overline{\Omega} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathcal{O}_\varepsilon \qquad \text{(Ishi 2000)}.$$

Proposition (Graczyk–Ishi)

(φ, E): any representation
$\exists 1\ \varepsilon = \varepsilon(\varphi) \in \{0,1\}^r$ s.t. φ is an ε-representation.

ε-representation

Calculation of $\varepsilon(\varphi)$

Put $d_{kj} := \dim V_{kj}$.

$$l^{(1)} := {}^t(\dim E_1, \ldots, \dim E_r) \qquad (E_j := \varphi(c_j)E),$$
$$l^{(k)} := \begin{cases} l^{(k-1)} - {}^t(0, \ldots, 0, d_{k,k-1}, \ldots, d_{r,k-1}) & (l^{(k-1)}_{k-1} > 0), \\ l^{(k-1)} & \text{(otherwise)}. \end{cases}$$

Then $\varepsilon(\varphi) = {}^t(\varepsilon_1, \ldots, \varepsilon_r)$ is defined by

$$\varepsilon_k = \begin{cases} 1 & (\text{if } l^{(k)}_k > 0), \\ 0 & \text{(otherwise)}. \end{cases}$$

Contents

Determination of α_j

Theorem

$V = \mathbb{R}c_1 \oplus E \oplus W$: clan of rank r with

W : subclan of V

(φ, E) : representation of $(W, \triangledown)$ $\quad (\varphi(w)\xi = \xi \triangledown w)$.

$\Delta_j(x)$: basic relative invariants of V $(j = 1, \ldots, r)$.

$$\Delta_j(\lambda c_1 + \xi + w) = \lambda^{-\alpha_j} \Delta_j^W(\lambda w - \tfrac{1}{2}Q[\xi]) \quad (j = 2, \ldots, r).$$

Let σ_W be the multiplier matrix of W and φ an ε-representation. Then

$$\begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix} = \sigma_W \begin{pmatrix} 1-\varepsilon_2 \\ \vdots \\ 1-\varepsilon_r \end{pmatrix} \quad \text{where } \varepsilon = \begin{pmatrix} \varepsilon_2 \\ \vdots \\ \varepsilon_r \end{pmatrix} \in \{0,1\}^{r-1}.$$

σ_V : multiplier matrix of V

$$\Rightarrow \sigma_V = \begin{pmatrix} 1 & 0 \\ \sigma_W \varepsilon & \sigma_W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_W \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon & I_{r-1} \end{pmatrix}.$$

Calculation of multiplier matrix

- For $k = 1, 2, \ldots, r-1$, we put

$$E^{[k]} := \bigoplus_{m>k} V_{mk}, \qquad V^{[k]} := \bigoplus_{k<l\leq k\leq r} V_{ml}.$$

$$\begin{pmatrix} & & \\ & \mathbb{R}c_k & {}^tE^{[k]} \\ & E^{[k]} & V^{[k]} \end{pmatrix}$$

- $V^{[k]}$ is a subclan of $V^{[k-1]}$.
- $(\mathcal{R}^{[k]}, E^{[k]})$ is a representation of $(V^{[k]}, \triangledown)$:

$$\mathcal{R}^{[k]}(x)\xi := \xi \triangledown x \quad (x \in V^{[k]},\ \xi \in E^{[k]}).$$

- Assume that $\mathcal{R}^{[k]}$ is an $\varepsilon^{[k]}$-representation ($\varepsilon^{[k]} \in \{0, 1\}^{r-k}$).

Calculation of multiplier matrix

Put $\mathcal{E}_k := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon^{[k]} & I_{r-k} \end{pmatrix}$.

Recalling

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_W \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon & I_{r-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_W \end{pmatrix} \mathcal{E}_1,$$

we obtain the following theorem.

Theorem

$$\sigma_V = \mathcal{E}_{r-1}\mathcal{E}_{r-2}\cdots\mathcal{E}_1.$$

Vinberg's polynomials

- $\|x\|^2 := \langle x \,|\, x \rangle$.
- Definition of Vinberg's polynomials $D_j(x)$ are as follows:
- Define $x^{(j)} = \sum_{k=j}^{r} x_{kk}^{(j)} c_k + \sum_{m>k\geq j} X_{mk}^{(j)} \in V^{[j-1]}$ by

$$x^{(1)} := x,$$
$$x_{kk}^{(j+1)} := x_{jj}^{(j)} x_{kk}^{(j)} - \frac{1}{2s_0(c_k)} \|X_{kj}^{(j)}\|^2 \quad (j < k \leq r),$$
$$X_{mk}^{(j+1)} := x_{jj}^{(j)} X_{mk}^{(j)} - X_{mj}^{(j)} \triangle X_{kj}^{(j)} \quad (j < k < m \leq r).$$

- Then
$$D_j(x) := x_{jj}^{(j)} \in \mathbb{R}.$$
- Note that each $D_j(x)$ is a relatively H-invariant polynomial.

Vinberg's polynomials

$D_1(x), \ldots, D_r(x)$ appear in the solution $h \in H$ of the equation

$$he_0 = x \quad (\text{given } x \in \Omega).$$

The diagonal components are calculated as

$$h_{11}^2 = D_1(x), \quad h_{jj}^2 = D_1(x)^{-1} \cdots D_{j-1}(x)^{-1} D_j(x) \quad (j \geq 2).$$

This implies

$$\begin{aligned} \Omega &= \{x \in V;\ D_1(x) > 0, \ldots, D_r(x) > 0\} \\ &= \{x \in V;\ \Delta_1(x) > 0, \ldots, \Delta_r(x) > 0\}. \end{aligned}$$

Recalling $\Delta_j(x)$ are described as

$$\Delta_j(he_0) = (h_{11})^{2\sigma_{j1}} \cdots (h_{jj})^{2\sigma_{jj}},$$

we obtain the main theorem.

Explicit expression

Main theorem

Let $\sigma_V = (\sigma_{jk})$ be the multiplier matrix of V. Then one has

$$\Delta_1(x) = D_1(x), \quad \Delta_j(x) = \frac{D_j(x)}{\prod_{i<j} D_i(x)^{\tau_{ji}}},$$

where $\tau_{ji} = -\sigma_{ji} + \sigma_{j,i+1} + \cdots + \sigma_{jj} \in \mathbb{Z}_{\geq 0}$.

To determine $\Delta_j(x)$:

Divide $D_j(x)$ by $\Delta_1(x), \ldots, \Delta_{j-1}(x)$ until not-divisible

$\downarrow$

Divide $D_j(x)$ by $D_i(x)$ τ_{ji}-times.

Thank you for your attention!